AN INFINITE DIMENSIONAL DESCRIPTOR SYSTEM MODEL FOR
ELECTRICAL CIRCUITS WITH TRANSMISSION LINES

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Abstract. In this paper a model of linear electrical circuits with transmission lines is derived. The equations obtained by the modified nodal analysis (MNA) are boundary-coupled with the telegraph equations who describe the behavior of the transmission lines. The resulting system of equations turns out to be an abstract differential-algebraic system and it is formulated as a descriptor system whose (generalized) state space is an infinite dimensional Hilbert space.

Key words. Analytic circuit theory, partial differential-algebraic equations, infinite dimensional linear system theory

Introduction. Nowadays, electrical circuits consist of a very large number ($\approx 10^7$) of components like resistors, capacitors, inductors, free and controlled voltage and current sources. Additionally, these circuits are operated in some higher frequency domains. As a consequence, several longer connections between components cannot be modelled as a short circuit anymore but rather as a transmission line (see [8]) The model of such a transmission line is shown in Figure 0.1.

\[ l \]

\[ L \]

\[ R \]

\[ L \]

\[ R \]

\[ C \]

\[ G \]

\[ l \]

\[ V(0, t) \]

\[ V(l, t) \]

\[ I(l, t) \]

\[ I(0, t) \]

\[ V(0, t) \]

\[ V(l, t) \]

\[ I(0, t) \]

\[ I(l, t) \]

The current and the voltage along the transmission line fulfill the telegraph equations:

\[ \frac{\partial}{\partial t} V(x, t) = -\frac{G}{C} V(x, t) - \frac{1}{C} \frac{\partial}{\partial x} I(x, t) \quad (0.1) \]

\[ \frac{\partial}{\partial t} I(x, t) = -\frac{1}{L} \frac{\partial}{\partial x} V(x, t) - \frac{R}{L} I(x, t), \quad x \in [0, l] \quad (0.2) \]

The aim of this work is to develop a model for electrical circuits containing several of these transmission lines. It is organized as follows: In the first section, the modified nodal analysis (MNA) for circuits with only lumped linear elements is presented as a descriptor system model. A state space model for the transmission lines is derived in the second section. The last section is about modelling linear circuits with lumped elements and transmission lines as a descriptor system while using the results of the first two sections.

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1. Circuits with Lumped Elements. The circuits are modelled as directed graph whose edges are weighted with the voltage-current relations of the electrical components (like e.g. Ohm’s law). See e.g. [9], [7] or [4] for details. Kirchhoff’s current law (KCL) says that the net current outflow vanishes at any vertex of the graph, i.e.

\[ A'i = 0, \]

where \( i \) is the vector containing the currents flowing through the circuit and \( A' \) is the reduced incidence matrix of the graph. According to Kirchhoff’s voltage law (KVL), every voltage can be derived from the node potential vector \( \phi \). In particular, we have

\[ u = A'^T \phi. \]

\( u \) is the vector having the edge voltages as components. One of the most powerful techniques in industrial circuit simulation is the so-called modified nodal analysis (MNA). There, the relations of the electrical components are split into a part that can be solved for \( i \) and a part that can be solved for \( u \). In the linear case, we have the form

\[
\left[
\begin{array}{cc}
I & P_1 \\
0 & P_2
\end{array}
\right]
\left[
\begin{array}{c}
i_1 \\
i_2
\end{array}
\right]
\left[
\begin{array}{cc}
Q_1 & 0 \\
Q_2 & I
\end{array}
\right]
\left[
\begin{array}{c}
u_1 \\
u_2
\end{array}
\right]
= \left[
\begin{array}{c}
c_1 \\
c_2
\end{array}
\right]
\]

for some coefficient matrices \( P_1, P_2, Q_1, Q_2 \). With the corresponding partition, we obtain from Kirchhoff’s laws

\[ A'_1i_1 + A'_2i_2 = 0 \quad \text{and} \quad \left[
\begin{array}{c}
u_1 \\
u_2
\end{array}
\right] = \left[
\begin{array}{c}
A'^T_1i_1 \\
A'^T_2i_2
\end{array}
\right] \phi. \]

This yields the MNA equations

\[
\left[
\begin{array}{cc}
A'_1Q_1 & A'^T_1P_1 - A'_2 \\
Q_2A'^T_1 + A'_2 & P_2
\end{array}
\right]
\left[
\begin{array}{c}
\phi \\
i_2
\end{array}
\right] = \left[
\begin{array}{c}
A'_1c_1 \\
c_2
\end{array}
\right]. \tag{1.1}
\]

Let now \( A' = (ARACALAIAV) \) be the incidence matrix generated by the graph of the given circuit. \( AR, AC, AL, AI, AV \) contain the resistive, capacitive, inductive branches and those of the current and voltage sources, respectively. Let \( i_R, i_C, i_L, i_I \) and \( i_V \) the corresponding current vectors. Using the partition

\[ i_1 = \begin{pmatrix} i_R \\ i_C \end{pmatrix}, \quad i_2 = \begin{pmatrix} i_L \\ i_I \\ i_V \end{pmatrix} \]

this leads to the following system of equations:

\[
ACCA'^T_C \frac{d}{dt} \phi + ARR^{-1}A'R\phi + ALi_L + Aviv + Aii_I = 0 \\
A'^T_V \phi - L \frac{d}{dt}i_L = 0 \\
A'^T_V \phi - u_V = 0.
\]

\( C, R \) and \( L \) are diagonal matrices whose entries are the capacities, resistances and inductances. The voltage sources are divided into the free and controlled ones, i.e.

\[
u_V = \nu_VA'^T\phi + \nu_CCA'^T_C \frac{d}{dt} \phi + \nu_Li_L + \nu_Ii_V + \nu_fu_f, \tag{1.2}
\]
where \( u_f \) represents the free voltages. \( \nu_V, \nu_C, \nu_L, \nu_{IV}, \nu_f \) are matrices which represent the amplifying gains of the controlled sources whose controlling variables are voltages and capacitive currents, inductive currents and currents of voltage sources. Since resistive currents depends algebraically on their voltages, the resistive-current-controlled voltage sources can be seen as voltage-controlled voltage sources.

In the same way, we have the relation

\[
A_i i_I = A_{IV} \mu_V A^T \phi + A_{IC} \mu C A^T_C \frac{d}{dt} \phi + A_{IL} \mu L I_L + A_{IV} \mu_{IV} \nu_{IV} + A_f \nu_f i_f, 
\]

where \( A_I = (A_{IV} A_{IC} A_{IL} A_{IV} A_f) \) and some amplifying gain matrices \( \mu_V, \mu_C, \mu_L, \mu_{IV}, \mu_f \).

Using the equations (1.2) and (1.3), we obtain the system

\[
E \dot{x} = Ax + Bu
\]

with

\[
x = \begin{pmatrix} \phi \\ i_L \\ i_V \end{pmatrix}, \quad u = \begin{pmatrix} u_f \\ i_f \end{pmatrix},
\]

\[
E = \begin{pmatrix} AC CA^T_C + A_{IC} \mu C A^T_C & 0 & 0 \\ 0 & L & 0 \\ \nu C CA^T_C & 0 & 0 \end{pmatrix},
\]

\[
A = \begin{pmatrix} -A_R L^{-1} A^T_R & -A_{IV} \mu V A^T & -A_L & -A_{IV} \mu_{IV} & 0 \\ A_{IL} & \nu_V A^T & \nu_L & -\nu_{IV} & 0 \\ 0 & -A_f & 0 \\ 0 & 0 & \nu_f & 0 \end{pmatrix},
\]

\[
B = \begin{pmatrix} 0 & -A_f \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Remark 1.1. Differential-algebraic control systems of the form \( E \dot{x} = Ax + Bu \) are called descriptor systems. They are e.g. treated in [2].

2. The Transmission Lines. In this section a state space model for transmission lines is derived. Let the equations (0.1) and (0.2) be given. We always assume without loss of generality that the length of the line is normalized, i.e. \( l = 1 \). This can be done by a transformation of the parameters. For convenience, we assume that the parameters \( C, L, G \) and \( R \) are constant along the line. As input, we choose \( \begin{pmatrix} u_{i_0}(t) \\ i_{i_1}(t) \end{pmatrix} := \begin{pmatrix} V(0, t) \\ I(1, t) \end{pmatrix} \) and output of the system is supposed to be \( \begin{pmatrix} u_{i_0}(t) \\ i_{i_1}(t) \end{pmatrix} := \begin{pmatrix} V(1, t) \\ I(0, t) \end{pmatrix} \).

The partial differential equations (0.1) and (0.2) can be formulated as an abstract ordinary differential equation in the function space \( L^2_2[0, 1] = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \in L^2_2[0, 1] \right\} \):

\[
\frac{d}{dt} \begin{pmatrix} V(x, t) \\ I(x, t) \end{pmatrix} = \begin{pmatrix} \frac{-G}{L} & 0 \\ -\frac{1}{L} & \frac{-1}{L} \end{pmatrix} \begin{pmatrix} V(x, t) \\ I(x, t) \end{pmatrix}
\]

with initial data

\[
\begin{pmatrix} V(\cdot, 0) \\ I(\cdot, 0) \end{pmatrix} = \begin{pmatrix} V_0 \\ I_0 \end{pmatrix} \in L^2_2[0, 1].
\]
and boundary conditions

\[
\begin{pmatrix}
V(0, t) \\
I(1, t)
\end{pmatrix} = \begin{pmatrix}
u T_0(t) \\
\nu T_1(t)
\end{pmatrix}. \tag{2.3}
\]

The differential operator

\[
\Gamma : D(\Gamma) \subset L^2_2[0, 1] \to L^2_2[0, 1]
\]

\[
\begin{pmatrix}
V(t) \\
I(t)
\end{pmatrix} \mapsto \begin{pmatrix}
\frac{\partial}{\partial x} V - \frac{1}{2} \frac{\partial}{\partial x} I \\
- \frac{1}{2} \frac{\partial}{\partial x} V - \frac{1}{2} I
\end{pmatrix}
\]

is unbounded and its domain is

\[
D(\Gamma) = \left\{ \begin{pmatrix}
V(t) \\
I(t)
\end{pmatrix} \in L^2_2[0, 1] \text{ with } V, I \text{ are absolutely continuous, and } \frac{\partial}{\partial x} V, \frac{\partial}{\partial x} I \in L^2[0, 1] \right\},
\]

which is dense in \( L^2_2[0, 1] \).

Altogether the equations (2.1), (2.2) and (2.3) can be reformulated as an abstract boundary control problem (see [1], p.121 ff.)

\[
\frac{d}{dt} \begin{pmatrix}
V(t) \\
I(t)
\end{pmatrix} = \Gamma \begin{pmatrix}
V(t) \\
I(t)
\end{pmatrix}
\]

\[
Pz(t) = u(t), \tag{2.4}
\]

where \( P : D(P) \to C^2, \begin{pmatrix}
V(t) \\
I(t)
\end{pmatrix} \mapsto \begin{pmatrix}
V(0) \\
I(1)
\end{pmatrix} \) is the so-called boundary operator and it is defined on

\[
D(P) = \left\{ \begin{pmatrix}
V(t) \\
I(t)
\end{pmatrix} \in L^2_2[0, 1] : V \text{ is continuous in } x = 0 \text{ and } I \text{ is continuous in } x = 1 \right\} \subset D(\Gamma)
\]

and has the kernel

\[
\ker(P) = \left\{ \begin{pmatrix}
V(t) \\
I(t)
\end{pmatrix} \in D(P) : V(0) = I(1) = 0 \right\}
\]

**Theorem 2.1.** The control system (2.4) is a boundary control system, i.e. the following assertions hold.

1. The operator \( A : D(A) \to L^2_2[0, 1]^2 \) with \( D(A) = D(P) \cap D(\Gamma) \) and

\[
Az = \Gamma z \quad \text{for } z \in D(A) \tag{2.5}
\]

is an infinitesimal generator of a \( C_0 \)-semigroup on \( L^2_2[0, 1] \).

2. There exists a \( B \in L(C^2, L^2_2[0, 1]) \) such that for all \( u \in C^2, Bu \in D(\Gamma) \), the operator \( \Delta B \) is an element of \( L(C^2, L^2_2[0, 1]) \) and

\[
PBu = u, \quad u \in C^2 \tag{2.6}
\]

**Proof.**

1. The fact that \( A \) generates a \( C_0 \)-semigroup on \( L^2_2[0, 1] \) can be proven using the Lumer-Philips-Theorem [6].
2. A right inverse $B : \mathbb{C}^2 \to \mathcal{L}_2^2[0,1]$ of $\Psi$ is e.g. given by

$$
B \begin{pmatrix} V_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} V_1 \cdot 1_{[0,1]} \\ I_2 \cdot 1_{[0,1]} \end{pmatrix},
$$

whereas $1_{[0,1]}$ is the constant function mapping to 1.

As state of the system we take $z(t) = \begin{pmatrix} V(\cdot,t) \\ I(\cdot,t) \end{pmatrix} - Bu(t)$. For weakly differentiable inputs $u$, it satisfies the following abstract differential equation in $\mathcal{L}_2^2[0,1]$ (see [1]).

$$
\frac{d}{dt} z(t) = Az(t) - B\dot{u}(t) + UBu(t), \\
\quad z_0 = \begin{pmatrix} V(0,\cdot) \\ I(0,\cdot) \end{pmatrix} - Bu(0).
$$

(2.7)

Since $A$ generates a $C_0$-semigroup, the unique solvability of the the initial value problem (2.7) is guaranteed.

The output $y$ is then determined by

$$
y(t) = Cz(t) + Du(t)
$$

with $C \begin{pmatrix} z_1(\cdot) \\ z_2(\cdot) \end{pmatrix} = \begin{pmatrix} z_1(1) \\ z_2(0) \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

**Remark 2.1.** $C : \mathcal{L}_2^2[0,1] \to \mathbb{C}^2$ is the output operator and not a capacity. In the rest of this paper it will be always clear from the context which of both is meant.

The equations 2.7 and 2.8 can also be rewritten in the following way:

$$
\frac{d}{dt} z = Az - \frac{d}{dt} BT_{10}u_{T0} + BT_{11}u_{T0} - \frac{d}{dt} BT_{21}i_{T1} + BT_{21}i_{T1}, \\
u_{T1} = CT_{1}z + u_{T0}, \\
i_{T0} = CT_{0}z + i_{T1},
$$

(2.9)

with $BT_{10} = \begin{pmatrix} 1_{[0,1]} \\ 0 \end{pmatrix}$, $BT_{20} = \begin{pmatrix} 0 \\ 1_{[0,1]} \end{pmatrix}$, $BT_{11} = \begin{pmatrix} -G \\ 0 \end{pmatrix}_{[0,1]}$, $BT_{21} = \begin{pmatrix} 0 \\ -R \end{pmatrix}_{[0,1]}$.

$CT_{1} \begin{pmatrix} V \\ I \end{pmatrix} = V(1)$ and $CT_{0} \begin{pmatrix} V \\ I \end{pmatrix} = I(0)$.

3. **The Setup of the Network Equations.** Let $A' = (ARACALAVAT0AT1AJ)$ be the reduced incidence matrix of the circuit. $AT_0$ and $AT_1$ include the branches of the left and right boundaries of the transmission lines and $i_{T0}, i_{T1}$ are the corresponding current vectors.

The MNA equations read then

$$
\frac{d}{dt} ACACAT_{\phi} + ARAR^{-1}AT_{\phi} \\
+ ALi_L + AVi_V + AT_0i_{T0} + AT_1i_{T1} + Ai_I = 0
$$

(3.1)

$$
AT_L\phi - L \frac{d}{dt} i_L = 0
$$

(3.2)

$$
AT_V\phi - u_V = 0.
$$

(3.3)
Now let the circuit have \( n_T \) transmission lines which are modelled as in the previous section with
\[
\begin{align*}
\dot{z}_i &= A_i z - \frac{d}{dt} B_{T10} u_{T0i} + B_{T11} u_{T0i} - \frac{d}{dt} B_{T21} i_{T1i} + B_{T21} i_{T1i} \\
u_{T0i} &= C_{T1i} z + u_{T0i} \\
i_{T0i} &= C_{T0i} z + i_{T1i} \quad \text{for } i = 1 \ldots n.
\end{align*}
\]

We define
\[
\begin{align*}
A_T &:= \text{diag}(A_1, \ldots, A_{n_T}), \\
B_{T10} &:= \text{diag}(B_{101}, \ldots, B_{10n_T}), \\
B_{T11} &:= \text{diag}(B_{111}, \ldots, B_{11n_T}), \\
B_{T20} &:= \text{diag}(B_{201}, \ldots, B_{20n_T}), \\
B_{T21} &:= \text{diag}(B_{211}, \ldots, B_{21n_T}), \\
C_{T0} &:= \text{diag}(C_{T01}, \ldots, C_{T0n_T}), \\
C_{T1} &:= \text{diag}(C_{T11}, \ldots, C_{T1n_T}) \quad \text{and}
\end{align*}
\]
\[
z := \begin{pmatrix} z_1 \\ \vdots \\ z_{n_T} \end{pmatrix}.
\]

Then we have
\[
\begin{align*}
\frac{d}{dt} z - A_T z - \frac{d}{dt} B_{T10} A_T^T \phi + B_{T20} A_T^T \phi - \frac{d}{dt} B_{T11} i_{T1} + B_{T21} i_{T1} &= 0 \quad (3.4) \\
C_T z + u_{T0} - u_T &= 0 \quad (3.5) \\
C_{T0} z + i_{T1} - i_{T0} &= 0 \quad (3.6)
\end{align*}
\]

The equations for the controlled sources are
\[
\begin{align*}
\nu V &= \nu V A_T^T \phi + \nu C A_T^T \frac{d}{dt} \phi + \nu L i_L + \nu V iv + \nu T0i_T0 + \nu T1i_T1 + \nu f u_f, \\
A_T i_i &= A_{IV} \mu V A_T^T \phi + A_{IC} \mu C A_T^T \frac{d}{dt} \phi + A_{IL} \mu L i_L + A_{IV} \mu v iv \\
&+ A_{IT0} \mu T0i_T0 + A_{IT1} \mu T1i_T1 + A_{f} \mu f i_f.
\end{align*}
\]

Plugging these two equations and the relation (3.6) into the equations (3.1) and (3.3), we obtain the following differential-algebraic system:
\[
E \dot{x} = Ax + Bu,
\]

with
\[
x = \begin{pmatrix} \phi \\ i_L \\ iv \\ i_{T1} \\ z \end{pmatrix}, \quad u = \begin{pmatrix} u_f \\ i_f \end{pmatrix}.
\]

The generalized state space of this system is then \( X = \mathbb{R}^{n_\phi + n_L + n_V + n_T} \times \mathcal{L}_2[0,1]^{2n_T} \), where \( n_\phi, n_L, n_V, n_T \) are the numbers of nodes, inductances, voltage sources and
transmission lines.

The operators $E$, $A$ and $B$ are given by

$$E = \begin{bmatrix} A_C C \tilde{A}_L^T + A_{IC} C \tilde{A}_L \quad 0 & 0 & 0 & 0 \\ 0 & \tilde{L} & 0 & 0 & 0 \\ 0 & 0 & \tilde{C} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -A_{I_J} \mu_{I_J} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -A_{R}^{-1} A_T^T & -A_L \mu_{L} & -A_V \mu_{V} & -A_{IT} \mu_{IT} \\ 0 & -v_L & -v_{IV} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with $E : X \to X$, $A : D(A) = \mathbb{R}^{n_L+n_V+n_T} \times D(A_1)^{n_T} \subset X \to X$, $B : \mathbb{R}^{n_{I_J}} \to X$, where $n_{I_J}$ and $n_{I_J}$ are the numbers of free voltage and current sources.

**Conclusion.** In this work, a generalized state space model for circuits with transmission lines and lumped linear elements was derived. The state space turned out to be an infinite dimensional Hilbert space. Such systems are also known as abstract differential-algebraic systems (ADAS) and are treated e.g. in [5]. There typical problems of differential algebraic equations like index concepts and consistent initialization are generalized to the infinite dimensional case and can be applied to the system presented in this paper.

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**REFERENCES**


Electric power transmission systems are the means of transmitting power from a generating source to various load centers (i.e. where the power is being used). Generating stations generate electrical power. These generating stations are not necessarily situated where the majority of the power is being consumed (i.e. the load center). Since distance is not the only factor that determines the ideal location for a generating station, the place where the power is generated may be quite far away from where it is used. Hence power transmission systems are crucial to the supply of power in electrical networks. Overall, electrical supply systems are the network through which consumers of electricity receive power from a generation source (such as a thermal power station). Performance modelling is the abstraction of a real system into a simplified representation to enable the prediction of performance. The creation of a model can provide insight into how a proposed or actual system will or does work. This can, however, point towards different things to people belonging to different fields of work. Performance modelling has many benefits, which includes: Relatively inexpensive prediction of future performance. Infinite Electrical Networks: A Reprise. A. h. zemanian, fellow, IEEE. Matmct--this is a t u t d p q m 011 d v c infinite electrid net- â€œWhat do you mean I canâ€™t do that,â€ retorted the. Â In this case, the appropriate model is an infinite RLC network [56]. There is a good reason to view infinite electrical net-works as practical models of important problems and, therefore, as comprisinga compelling research area. To be sure, the jump in complexity from finite networks to infinite ones is comparable to thejump in complexity from finitsdimensional spaces to infinite-dimensional spaces. On the other hand, the theory of infinite electrical net-works is still in its puberty with many questions largely, unexplored, especially with regard to computational prob-lems. A Descriptor System Model for Electrical Circuits with Transmission Lines. 3. where uf represents the free voltages. Ì½V , Ì½C , Ì½L, Ì½IV , Ì½f are matrices which rep-resent the amplifying gains of the controlled sources whose controlling variables are voltages and capacitive currents, inductive currents and currents of voltage sources. (1.3). where AI = (AIV AIC AILAIiV Afi) and some amplifying gain matrices AµV , AµC , AµL, AµiV , Aµf. Using the equations (1.2) and (1.3), we obtain the system Ex˙ = Ax + Bu with. x E. = = iAi†VLCC,ATCu+=0AICuiAµf C. In this paper a model of linear electrical circuits with transmission lines is de-rived. The equations obtained by the modified nodal analysis (MNA) are boundary-coupled with the telegraph equations who describe the behavior of the transmission lines. The resulting system of equations turns out to be an abstract differential-algebraic system and it is formulated as a descriptor system whose (generalized) state space is an infinite dimensional Hilbert space. Introduction.Â The last section is ab out modelling linear circuits with lumped, elements and transmission lines as a descriptor system while using the results of the. ï¬rst two sections. â—- Fachbereich Mathematik, UniversitÄ“at Kaiserslautern, (reis@mathematik.uni-kl.de).