Boundedness for some Schrödinger type operators on Morrey spaces related to certain nonnegative potentials

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\section{Introduction}

In this paper, we consider the Schrödinger differential operator

\[ \mathcal{L} = -\Delta + V(x) \quad \text{on} \quad \mathbb{R}^n, \quad n \geq 3, \]

where \( V(x) \) is a nonnegative potential belonging to the reverse Hölder class \( B_q \) for \( q \geq n/2 \).

A nonnegative locally \( L^1 \) integrable function \( V(x) \) on \( \mathbb{R}^n \) is said to belong to \( B_q \) \( (q > 1) \) if there exists \( C > 0 \) such that the reverse Hölder inequality

\[ \left( \frac{1}{|B|} \int_B V^q \, dx \right)^{1/q} \leq C \left( \frac{1}{|B|} \int_B V \, dx \right) \]

holds for every ball \( B \) in \( \mathbb{R}^n \); see [10].

For \( x \in \mathbb{R}^n \), the function \( m_V(x) \) is defined by

\[ \frac{1}{m_V(x)} = \sup_{r > 0} \left\{ \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) \, dy \leq 1 \right\}. \]

Let \( p \in [1, \infty) \), \( \alpha \in (-\infty, \infty) \) and \( \lambda \in [0, n) \). For \( f \in L^p_{\text{loc}}(\mathbb{R}^n) \) and \( V \in B_q \) \( (q > 1) \), we say \( f \in L^p_{\alpha, V}(\mathbb{R}^n) \) \( (\text{Morrey spaces related to the nonnegative potential} \ V) \) provided that

\[ \left\| f \right\|_{L^p_{\alpha, V}(\mathbb{R}^n)} = \sup_{B(x_0, r) \subset \mathbb{R}^n} \left( 1 + rm_V(x_0) \right)^{\alpha} r^{-\lambda} \int_{B(x_0, r)} \left| f(x) \right|^p \, dx < \infty. \]

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where $B = B(x_0, r)$ denotes a ball with centered at $x_0$ and radius $r$. In particular, when $\alpha = 0$ or $V = 0$ and $0 < \lambda < n$, the space $L^{p, \lambda}_{\alpha, V}(\mathbb{R}^n)$ is the class Morrey space $L^{p, \lambda}(\mathbb{R}^n)$ (see [8]), some new properties of $L^{p, \lambda}(\mathbb{R}^n)$ have been studied in [1,6,12]). It is easy to see that $L^{p, \lambda}_{\alpha, V}(\mathbb{R}^n) \subset L^{p, \lambda}(\mathbb{R}^n)$ for $\alpha > 0$, and $L^{p, \lambda}(\mathbb{R}^n) \subset L^{p, \lambda}_{\alpha, V}(\mathbb{R}^n)$ for $\alpha < 0$.

Shen [10] showed the Schrödinger type operators $\nabla(-\Delta + V)^{-1}\nabla$, $\nabla(-\Delta + V)^{-1/2}$ and $(-\Delta + V)^{-1/2}\nabla$ are the standard Calderón–Zygmund operators provided that $V \in B_n$. In particular, the kernels $K$ of above operators satisfy the following inequality

$$|K(x, y)| \leq \frac{C_k}{(1 + |x - y|)k} \frac{1}{|x - y|^n}$$

for any $k \in \mathbb{N}$.

From [10], we know that there exists a nonnegative potential $V \in B_q$ with $q < n$ so that these operators $\nabla(-\Delta + V)^{-1}\nabla$, $\nabla(-\Delta + V)^{-1/2}$ and $(-\Delta + V)^{-1/2}\nabla$ may not be bounded on $L^p$ for all $1 < p < \infty$. Hence, in the rest of this paper, we always assume that $T$ is one of the Schrödinger type operators $\nabla(-\Delta + V)^{-1}\nabla$, $\nabla(-\Delta + V)^{-1/2}$ and $(-\Delta + V)^{-1/2}\nabla$ with $V \in B_n$.

It is well known that the boundedness of the standard Calderón–Zygmund operators and their commutators have been established on the class Morrey spaces (see [5]). Hence, it will be an interesting question whether we can establish the boundedness of Schrödinger type operators on the Morrey spaces related to certain nonnegative potentials. The main purpose of this paper is to answer the above question. More precisely, we obtain the following results.

**Theorem 1.1.** Suppose $\alpha \in (-\infty, \infty)$ and $\lambda \in (0, n)$.

(i) If $1 < p < \infty$, then

$$\|Tf\|_{L^p_{\alpha, V}(\mathbb{R}^n)} \leq C \|f\|_{L^p_{\alpha, V}(\mathbb{R}^n)},$$

where $C$ is independent of $f$.

(ii) If $p = 1$, then for any $t > 0$,

$$t(1 + rm_{\alpha, V}(x))^{-\alpha} \|\{y \in B(x, r): |Tf(y)| > t\}\| \leq C \|f\|_{L^1_{\alpha, V}(\mathbb{R}^n)}$$

holds for all balls $B$, where $C$ is independent of $x$, $r$, $t$ and $f$.

Let $b \in BMO$ (see its definition in [11]), we define the commutator of $T$ by

$$[b, T]f = bTf - T(bf).$$

**Theorem 1.2.** Suppose $b \in BMO$, $\alpha \in (-\infty, \infty)$ and $\lambda \in (0, n)$.

(i) If $1 < p < \infty$, then

$$\|[b, T]f\|_{L^p_{\alpha, V}(\mathbb{R}^n)} \leq C \|f\|_{L^p_{\alpha, V}(\mathbb{R}^n)},$$

where $C$ is independent of $f$.

(ii) If $p = 1$, then for any $t > 0$,

$$r^{-\lambda}(1 + rm_{\alpha, V}(x))^{-\alpha} \|\{y \in B(x, r): [b, T]f(y) > t\}\| \leq C \sup_{B(x, r) \subset \mathbb{R}^n} \left(1 + rm_{\alpha, V}(x)\right)^{-\lambda} \int_{B(x, r)} \frac{|f(y)|}{t} \ln \left(2 + \frac{|f(y)|}{t}\right) dy$$

holds for all balls $B$, where $C$ is independent of $x$, $r$, $t$ and $f$.

Next, we consider the boundedness of fractional integrals related to Schrödinger operators.

Let $\mathcal{L} = -\Delta + V$ with $V \in B_q$ for $q > n/2$ and its associated semigroup:

$$T_tf(x) = e^{-t\mathcal{L}}f(x) = \int_{\mathbb{R}^n} k_t(x, y) f(y) dy, \quad f \in L^2(\mathbb{R}^n), \quad t > 0.$$  \hspace{1cm} (1.2)

The $\mathcal{L}$-fractional integral operator is defined by

$$I_{\beta}f(x) = \mathcal{L}^{-\beta/2}f(x) = \int_0^\infty e^{-t\mathcal{L}} f(x) dt$$

for $0 < \beta < n$. 


Theorem 1.3. Suppose $V \in B_{n/2}$, $\alpha \in (-\infty, \infty)$ and $0 < \beta < n$.

(i) If $1 < p < n/\beta$, $1/q = 1/\beta - n/\beta$, $\theta = q/p$ and $0 < \lambda < n/\theta$, then
\[ \|T_{\beta}f\|_{L^{q,\theta}_{\lambda,\beta}(\mathbb{R}^n)} \leq C\|f\|_{L^{p,\lambda}_{\beta,\theta}(\mathbb{R}^n)}, \]
where $C$ is independent of $f$.

(ii) If $p = 1$ and $q = n/(n - \beta)$, then for any $t > 0$,
\[ t(1 + rm_{V}(x))^\alpha \left( \int_{B(x, r)} |T_{\beta}f(y)|^q \right)^{1/q} \leq C r^\lambda \|f\|_{L^{p,\lambda}_{\beta,\theta}(\mathbb{R}^n)} \]
holds for all balls $B$, where $C$ is independent of $x, r, t$ and $f$.

Let $b \in \text{BMO}$, we define the commutator of $T_{\beta}$ by
\[ [b, T_{\beta}]f = bT_{\beta}f - T_{\beta}(bf). \]

Theorem 1.4. Let $b \in \text{BMO}$, $V \in B_{n/2}$, $\alpha \in (-\infty, \infty)$ and $0 < \beta < n$.

(i) If $1 < p < n/\beta$, $1/q = 1/\beta - n/\beta$, $\theta = q/p$ and $0 < \lambda < n/\theta$, then
\[ \|[b, T_{\beta}]f\|_{L^{q,\theta}_{\lambda,\beta}(\mathbb{R}^n)} \leq C\|f\|_{L^{p,\lambda}_{\beta,\theta}(\mathbb{R}^n)}, \]
where $C$ is independent of $f$.

(ii) If $p = 1$ and $q = n/(n - \beta)$, then for any $t > 0$,
\[ r^{-\lambda}(1 + rm_{V}(x))^\alpha \left( \int_{B(x, r)} |[b, T_{\beta}]f(y)|^q \right)^{1/q} \leq C \sup_{B(x, r) \subset \mathbb{R}^n} (1 + rm_{V}(x))^\alpha r^{-\lambda} \Phi(\int_{B(x, r)} \frac{|f(y)|}{t} \ln(2 + \frac{|f(y)|}{t}) dy) \]
holds for all balls $B$, where $\Phi(t) = [t \log(2 + t^{\beta/n})^{n/(n - \beta)}]$, and $C$ is independent of $x, r, t$ and $f$.

We remark that even in the classical Morrey space, the above results about the case $p = 1$ in Theorems 1.2 and 1.4 are also new; see [5].

Throughout this paper, $C$ is a positive constant which is independent of the main parameters and not necessary the same at each occurrence.

2. Proof of Theorems 1.1 and 1.2

We first introduce some notations and recall some properties of the auxiliary function $m_{V}(x)$. We need the following lemma about $m_{V}(x)$.

Lemma 2.1. (See [10].) Suppose $V \in B_{q}$ with $q \geq n/2$. Then there exist positive constants $C$ and $k_{0}$ such that

(a) $m_{V}(x) \sim m_{V}(y)$ if $|x - y| \leq C m_{V}(x)$,

(b) $m_{V}(y) \leq C(1 + |x - y|m_{V}(x))^{k_{0}}m_{V}(x)$,

(c) $m_{V}(y) \geq \frac{Cm_{V}(x)}{(1 + |x - y|m_{V}(x))^{k_{0}/(n + \beta)}}$.

Proof of Theorem 1.1. Without loss of generality, we may assume that $\alpha < 0$. Pick any $x_{0} \in \mathbb{R}^n$ and $r > 0$, and write
\[ f(x) = f_{0}(x) + \sum_{i=1}^{\infty} f_{i}(x), \]
where $f_{0} = \chi_{B(x_{0}, 2r)} f$, $f_{i} = \chi_{B(x_{0}, 2^{i+1}r), B(x_{0}, 2^{i}r)} f$ for $i \geq 1$. Hence, we have
\[ \left( \int_{B(x_{0}, r)} |Tf_{0}(x)|^{p} dx \right)^{1/p} \leq \left( \int_{B(x_{0}, r)} |Tf_{0}(x)|^{p} dx \right)^{1/p} + \sum_{i=1}^{\infty} \left( \int_{B(x_{0}, r)} |Tf_{i}(x)|^{p} dx \right)^{1/p}. \tag{2.1} \]
By the $L^{p}$ boundedness of $T$, we obtain
\[ \int_{B(x_{0}, r)} |Tf_{0}(x)|^{p} dx \leq C(1 + rm_{V}(x_{0}))^{-\alpha} \|f\|_{L^{p,\lambda}_{\beta,\theta}(\mathbb{R}^n)}^{p}, \]
\[ \int_{B(x_{0}, r)} |Tf_{0}(x)|^{p} dx \leq C(1 + rm_{V}(x_{0}))^{-\alpha} \|f\|_{L^{p,\lambda}_{\beta,\theta}(\mathbb{R}^n)}^{p}, \tag{2.2} \]
By Lemma 2.1 and the John–Nirenberg inequality on BMO (see [7]), as well as (1.1), we have
\[
\int_{B(x_0, r)} |T f_i(x)|^p \, dx \leq C \int_{B(x_0, 2^{-1}r)} \left( \int_{B(x_0, 2^{i+1}r) \setminus B(x_0, 2^i r)} |K(x, y) f(y)| \, dy \right)^p \, dx
\]
\[
\leq C_k \int_{B(x_0, r)} \frac{1}{(1 + 2^i r \nu (x_0) ^k)^p} \, dx \times \int_{B(x_0, 2^{i+1}r)} |f(y)|^p \, dy \, dx
\leq C_k (2^i r)^{-n} \int_{B(x_0, r)} \left( \frac{1 + 2^i r \nu (x_0) ^k} {1 + 2^{i+1} r \nu (x_0) ^k + 1} \right)^p \, dx \| f \|_{L_p^{k, \lambda} (\mathbb{R}^n)}^p
\leq C_k (2^i r)^{-n} \int_{B(x_0, r)} \left( \frac{1 + 2^i r \nu (x_0) ^k} {1 + 2^{i+1} r \nu (x_0) ^k + 1} \right)^p \, dx \| f \|_{L_p^{k, \lambda} (\mathbb{R}^n)}^p
\leq C_k (2^i r)^{-n} \int_{B(x_0, r)} \left( \frac{1 + 2^i r \nu (x_0) ^k} {1 + 2^{i+1} r \nu (x_0) ^k + 1} \right)^p \, dx \| f \|_{L_p^{k, \lambda} (\mathbb{R}^n)}^p.
\]
(2.3)

From (2.1)–(2.3) with \( k = (-\lceil \alpha \rceil + 1)(k_0 + 1) \), we obtain
\[
\| T f \|_{L_p^{k, \lambda} (\mathbb{R}^n)} \leq C \| f \|_{L_p^{k, \lambda} (\mathbb{R}^n)}.
\]
As for the case \( p = 1 \) the proof can be given by replacing (2.2) with the corresponding weak estimate.

**Proof of Theorem 1.2.** Without loss of generality, we may assume that \( \alpha < 0 \). Pick any \( x_0 \in \mathbb{R}^n \) and \( r > 0 \), as in the proof of Theorem 1.1, we write
\[
T f(x) = f_0(x) + \sum_{i=1}^{\infty} f_i(x).
\]
By the \( L^p \) boundedness of \( [b, T] \), we get
\[
\int_{B(x_0, r)} |[b, T] f_0(x)|^p \, dx \leq C + \nu (x_0) ^k \| f \|_{L_p^{k, \lambda} (\mathbb{R}^n)}^p.
\]
Set
\[
b_r = \frac{1} {B(x_0, r)} \int_{B(x_0, r)} b(x) \, dx.
\]
When \( i \geq 1 \), by Lemma 2.1 and the John–Nirenberg inequality, we have
\[
\left( \int_{B(x_0, r)} \left| [b, T] f_i(x) \right|^p \, dx \right)^{1/p}
\leq \frac{C_k} {1 + \nu (x_0) ^k} \cdot 2^{-in}
\times \left\{ \left( \int_{B(x_0, r)} |b(x) - b_r|^p \, dx \right)^{1/p} \int_{B(x_0, 2^{i+1}r)} |f(y)|^p \, dy + r^{n/p} \int_{B(x_0, 2^{i+1}r)} |f(y) - b_r| |f(y)| \, dy \right\}
\leq C_k \| b \|_{BMO} (2^i r)^{-n} \int_{B(x_0, r)} \left( \frac{1 + 2^i r \nu (x_0) ^k + \alpha / p} {1 + 2^{i+1} r \nu (x_0) ^k + \alpha / p} \right)^p \, dx \| f \|_{L_p^{k, \lambda} (\mathbb{R}^n)}^p.
\]
(2.4)
If take \( k = (-\lceil \alpha \rceil + 1)(k_0 + 1) \) in (2.4), then we obtain
\[
\| [b, T] f \|_{L_p^{k, \lambda} (\mathbb{R}^n)} \leq C \| f \|_{L_p^{k, \lambda} (\mathbb{R}^n)}.
\]
It remains to consider the case \( p = 1 \). From [9], we know that for any \( t > 0 \)
\[
\left| \{ x \in \mathbb{R}^n : |[b, T] f(y)| > t \} \right| \leq C \int_{\mathbb{R}^n} \frac{|f(x)|} {t} \ln \left( 2 + \frac{|f(x)|} {t} \right) \, dx.
\]
From this, we have
\[ r^{-\lambda}(1 + rm_V(x_0))^\alpha \left| \left\{ y \in B(x_0, r): \|b, T|f_0(y)| > t \right\} \right| \leq C(1 + rm_V(x_0))^\alpha r^{-\lambda} \int_{B(x_0, r)} \frac{|f(x)|}{t} \ln \left( 2 + \frac{|f(x)|}{t} \right) \, dx. \] (2.5)

Set
\[ b_{2^{i+1}r} = \frac{1}{|B(x_0, 2^{i+1}r)|} \int_{B(x_0, 2^{i+1}r)} b(x) \, dx. \]

When \( i \geq 1 \), by Lemma 2.1 with \( k = (-\alpha + 1)(k_0 + 1) \) and the John–Nirenberg inequality, we get
\[ r^{-\lambda} r^{1 - \lambda}(1 + rm_V(x)) \|f\|_{B(x_0, r)} \|f\|_{B(x_0, r)} \left| \left\{ y \in B(x_0, r): \|b, T|f_0(y)| > t \right\} \right| \leq C \left( 1 + 2^{i} rm_V(x) \right)^\alpha 2^{-i n - \lambda} \int_{B(x_0, r)} \left| f(y) \right| \, dy \]
where in the second and third inequality, we used the following facts (see [9]):
\[ \|f\|_{L \log L, B} = \inf \left\{ \lambda > 0: \frac{1}{\lambda} \int_{B} \left| f(y) \right| \log \left( 2 + \frac{|f(y)|}{\lambda} \right) \, dy \leq 10 \right\} \]
and
\[ \|f\|_{\exp L, B} = \inf \left\{ \lambda > 0: \frac{1}{\lambda} \int_{B} \exp \left( \frac{|f(y)|}{\lambda} \right) \, dy \leq 10 \right\} \]
the generalized Hölder inequality
\[ \frac{1}{|B|} \int_{B} \left| f(y) h(y) \right| \, dy \leq C \|f\|_{L \log L, B} \|h\|_{\exp L, B} \]
and
\[ \|f\|_{L \log L, B} \leq \inf_{w > 0} \left\{ w + \frac{w}{|B|} \int_{B} \left| f(y) \right| \log \left( 2 + \frac{|f(y)|}{w} \right) \, dy \right\} \leq 2 \|f\|_{L \log L, B}. \]

Combining (2.5) and (2.6), we obtain that
\[ r^{-\lambda}(1 + rm_V(x))^\alpha \left| \left\{ y \in B(x, r): \|b, T|f(y)| > t \right\} \right| \leq C \sup_{B(x, r) \subset \mathbb{R}^n} (1 + rm_V(x))^\alpha r^{-\lambda} \int_{B(x, r)} \frac{|f(y)|}{t} \ln \left( 2 + \frac{|f(y)|}{t} \right) \, dy \]
holds for all balls \( B \), where \( C \) is independent of \( x, r, t \) and \( f \).

Thus, Theorem 1.2 is proved. \( \Box \)
3. Proof of Theorems 1.3 and 1.4

We first need the following lemma.

**Lemma 3.1.** (See [4].) Let \( k_t(x, y) \) be as in (1.2). For every nonnegative integer \( k \), there is a constant \( C_k \) such that

\[
0 \leq k_t(x, y) \leq C_k t^{-n/2} \exp\left(-\frac{|x-y|^2}{5t}\right) \left(1 + \sqrt{m_V(x)} + \sqrt{m_V(y)}\right)^{-k}.
\]

**Proof of Theorem 1.3.** Without loss of generality, we may assume that \( \alpha < 0 \). Pick any \( x_0 \in \mathbb{R}^n \) and \( r > 0 \), as the proof of Theorem 1.1, we write

\[
f(x) = f_0(x) + \sum_{i=1}^{\infty} f_i(x).
\]

Hence, we have

\[
\left( \int_{B(x_0, r)} |I_{\beta} f(x)|^q \, dx \right)^{1/q} \leq \left( \int_{B(x_0, r)} |I_{\beta} f_0(x)|^q \, dx \right)^{1/q} + \sum_{i=1}^{\infty} \left( \int_{B(x_0, r)} |I_{\beta} f_i(x)|^q \, dx \right)^{1/q}.
\]

Let \( \theta = q/p \). By the \( L^p - L^q \) boundedness of \( I_{\beta} \) (see [11]), we get

\[
\int_{B(x_0, r)} |I_{\beta} f_0(x)|^q \, dx \leq C (1 + rm_V(x_0))^\alpha \| f \|^q_{L_\alpha^p (\mathbb{R}^n)}.
\]

For \( x \in B(x_0, r) \) and \( y \in \mathbb{R}^n \setminus B(x_0, 2r) \), we claim that for any \( N \in \mathbb{N} \), there exists a \( C_N \) such that

\[
\int_0^\infty t^{-(n-\beta)/2-1} k_t(x, y) \, dt \leq C_N \frac{1}{(1 + |x_0 - y| m_V(x_0))^N |x_0 - y|^{n-\beta}}.
\]

In fact, let \( r_0 = 1/m_V(x_0) \) and \( r_1 = |x_0 - y| \). Without loss of generality, we may assume \( r_0 \leq r_1 \), otherwise, (3.1) holds obviously. Then

\[
\int_0^\infty t^{\beta/2-1} k_t(x, y) \, dt = \int_0^{|y-x_0|^2} t^{\beta/2-1} k_t(x, y) \, dt + \int_{|y-x_0|^2}^\infty t^{\beta/2-1} k_t(x, y) \, dt
\]

\[
= I + II.
\]

For \( II \), by Lemmas 3.1 and 2.1, we have

\[
II \leq C_k \int_{|y-x_0|^2}^\infty t^{-(n-\beta)/2-1} \exp\left(-\frac{|x-y|^2}{5t}\right) (1 + \sqrt{m_V(x)})^{-k} \, dt
\]

\[
\leq C_k (1 + |y-x_0| m_V(x))^{-k} \int_{|y-x_0|^2}^\infty t^{-(n-\beta)/2-1} \, dt
\]

\[
\leq C_k (1 + |y-x_0| m_V(x))^{-k/|y-x_0|^2} |y-x_0|^\beta-n
\]

\[
\leq C_N (1 + |y-x_0| m_V(x))^{-k/|y-x_0|^2} |y-x_0|^\beta-n,
\]

taking \( N = [k/|y-x_0|^2] \).

For \( I \), by Lemmas 3.1 and 2.1 again, we have

\[
I \leq C \int_0^{r_0^2} t^{-(n-\beta)/2-1} \exp\left(-\frac{|x-y|^2}{5t}\right) \, dt + C \int_{r_0^2}^{r_1^2} t^{-(n-\beta)/2-1} \exp\left(-\frac{|x-y|^2}{5t}\right) \, dt
\]

\[
\leq C r_1^{\beta-n} \int_{r_1^2/r_0^2} t^{-(n-\beta)/2-1} \exp(-t/20) \, dt + C (r_0)^{-(n-\beta)-2} r_1^2 \exp\left(-\frac{r_1^2}{20r_0^2}\right).
\]
\[
\leq C r_1^{-\alpha} \exp\left(-\frac{r_1^2}{40r_0^2}\right) + C (r_0)^{-(\alpha-\beta)-2} r_1^2 \exp\left(-\frac{r_1^2}{20r_0^2}\right)
\]
\[
\leq C_N (1 + |y - x_0|^n V(x_0))^{-N} |y - x_0|^{\beta - n}.
\]

Thus (3.1) is proved.

From (3.1), we obtain
\[
\left\{ \begin{array}{l}
\int_{B(x_0, r)} |T \beta f_i(x)|^q \, dx \\
\int_{B(x_0, r)} \left( \int_{B(x_0, 2r)} |T \beta f_i(x)|^q \, dx \right)^{\frac{1}{q}} \\
\leq C_N \left( \int_{B(x_0, r)} \frac{2^{i(r)} - (\alpha/p - \beta)q}{(1 + 2i r m V(x_0))^{Nq}} \left( \int_{B(x_0, 2r)} |f(y)|^q \, dy \right)^{\frac{q}{p}} \, dx \right)^{\frac{1}{q}} \\
\leq C_N (2^{i(r)})^{\beta - n} \int_{B(x_0, r)} \frac{1}{(1 + 2i r m V(x_0))^{(N + \alpha/p)q}} \, dx \| f \|_{L_{p, \alpha}^p (\mathbb{R}^n)}^q \\
\end{array} \right.
\]

where \( N = -[\alpha] + 1. \)

Note that \( \theta \lambda < n. \) So
\[
\| T \beta f \|_{L_{p, \alpha}^p (\mathbb{R}^n)} \leq C \| f \|_{L_{p, \alpha}^p (\mathbb{R}^n)}.
\]

As for the case \( p = 1 \) the proof is similar. \( \square \)

**Proof of Theorem 1.4.** Without loss of generality, we may assume that \( \alpha < 0. \) Pick any \( x_0 \in \mathbb{R}^n \) and \( r > 0, \) as in the proof of Theorem 1.1, we write
\[
f(x) = f_0(x) + \sum_{i=1}^{\infty} f_i(x).
\]

Let \( \theta = q/p. \) By the \( L^p \) boundedness of \([b, T]\), we get
\[
\int_{B(x_0, r)} \| [b, T] f_0(x) \|^q \, dx \leq C \left( 1 + r m V(x_0) \right)^{-\theta q} \| f \|_{L_{p, \alpha}^p (\mathbb{R}^n)}^q.
\]

Set
\[
b_r = \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} b(x) \, dx.
\]

When \( i \geq 1, \) by (3.1) and the John–Nirenberg inequality, we have
\[
\left( \int_{B(x_0, r)} \| [b, T] f_i(x) \|^q \, dx \right)^{1/q} \leq \frac{C_N}{1 + 2i r m V(x_0))^{Nq}} (2^{i(r)})^{-(\alpha - \beta)}
\]
\[
\times \left\{ \left( \int_{B(x_0, r)} |b(x)|^q \, dx \right)^{1/q} \int_{B(x_0, 2r)} \| f(y) \| \, dy + r^{n/q} \int_{B(x_0, 2r)} |b(y) - b_r| \| f(y) \| \, dy \right\}
\]
\[
\leq C_N \| b \|_{BMO} 2^{i(r/\alpha - n/q)} \| f \|_{L_{p, \alpha}^p (\mathbb{R}^n)}^{2^{i(r/\alpha - n/q)} / \theta q}
\]
\[
\leq C_N \| b \|_{BMO} 2^{i(r/\alpha - n/q)} \| f \|_{L_{p, \alpha}^p (\mathbb{R}^n)},
\]

taking \( N = -[\alpha] + 1 \) in the last inequality.

Then,
\[
\| [b, T] f \|_{L_{p, \alpha}^p (\mathbb{R}^n)} \leq C \| f \|_{L_{p, \alpha}^p (\mathbb{R}^n)}.
\]

It remains to consider the case \( p = 1 \) and \( q = n/(n - \beta). \) From [2] and [3], we know
\[
\| y \in \mathbb{R}^n : [b, T] f(y) > t \| \leq C \Phi \left( \int_{\mathbb{R}^n} \frac{|f(x)|}{t} \ln \left( 2 + \frac{|f(x)|}{t} \right) \, dx \right),
\]
where \( \Phi(t) = [t \log(2 + t^{\beta/(n)})]^{n/(n - \beta)}. \)
From this, we have
\[ r^{-\lambda} (1 + r_m V(x_0))^\alpha \left[ \left\{ y \in B(x_0, r) : \| b \cdot \mathcal{I}_f \| f_0(y) > t \right\} \right] \]
\[ \leq C \sup_{B(x_0, r) \subset \mathbb{R}^n} (1 + r_m V(x_0))^{\alpha} r^{-\lambda} \Phi \left( \int_{B(x_0, r)} \frac{|f(x)|}{t} \ln \left( 2 + \frac{|f(x)|}{t} \right) dx \right). \]  
(3.2)

Set
\[ b_{2i+1} = \frac{1}{|B(x_0, 2^{i+1} r)|} \int_{B(x_0, 2^{i+1} r)} b(x) \, dx. \]

When \( i \geq 1 \), by (3.1) with \( N = [\alpha] + 1 \), we obtain
\[ r^{-\lambda} t^{-q} (1 + t_0 m V(x_0))^{\alpha} \int_{B(x_0, r)} \| b \cdot \mathcal{I}_f \| f_i(y) \| f(y) \| dy \]
\[ \leq C_N \left( 1 + 2^i r_m V(x_0) \right)^{Nq} (2^i r)^{-\lambda \alpha} (1 + r_m V(x_0))^{\alpha} t^{-q} \]
\[ \times \left\{ \int_{B(x_0, r)} |b(x) - b_{2i+1}|^q \, dx \left( \int_{B(x_0, 2^{i+1} r)} |f(y)|^q \, dy \right)^{\frac{q}{r}} + r^{- \lambda} \left( \int_{B(x_0, 2^{i+1} r)} |b(y) - b_{2i+1}|^q \, dy \right)^{\frac{q}{r}} \right\}. \]

By (3.2) and (3.3), we obtain the desired result. This completes the proof. \( \square \)

4. The Calderón–Zygmund inequality

For the open set \( \Omega \subset \mathbb{R}^n \) and \( V \in B_n \), we say \( f \in L^{p, \lambda}_{\alpha, V}(\Omega) \) if
\[ \| f \|_{p, \lambda}_{\alpha, V}(\Omega) = \sup_{B(x_0, r) \subset \mathbb{R}^n} (1 + r_m V(x_0))^{\alpha} r^{- \lambda} \int_{B(x_0, r) \cap \Omega} |f(x)|^p \, dx < \infty. \]

In this section, we consider the behavior of the solution of the following Schrödinger equation
\[ (-\Delta + V) u = f(x), \text{ a.e. } x \in \Omega, \]
where \( f \in L^{p, \lambda}_{\alpha, V}(\Omega) \), \( 1 < p < \infty \), \( 0 < \lambda < n \) and \( \alpha \in (-\infty, \infty) \).
Theorem 4.1. Let $\Omega$ be an open set in $\mathbb{R}^n$ and $\alpha \in (-\infty, \infty)$. If $f \in L_{\alpha,V}^{p,\lambda}(\Omega)$, then there exists a function $u \in L_{\alpha,V}^{2,\theta}(\Omega)$, where $1 < p < n/2$, $1/p - 1/q = 2/n$, $\theta = q/p$ and $0 < \lambda < n/\theta$ such that

$$(-\Delta + V)u = f(x), \quad \text{a.e. } x \in \Omega.$$ 

Furthermore,

$$\|D^2u\|_{L_{\alpha,V}^{p,\lambda}(\Omega)} \leq C \|f\|_{L_{\alpha,V}^{p,\lambda}(\Omega)}, \quad (4.1)$$

where $1 < p < \infty$ and $0 < \lambda < n$;

$$\|Du\|_{L_{\alpha,V}^{\infty,\lambda}(\Omega)} \leq C \|f\|_{L_{\alpha,V}^{p,\lambda}(\Omega)}, \quad (4.2)$$

where $1 < p < n$, $1/p - 1/q = 1/n$, $\theta_1 = q/p$ and $0 < \lambda < n/\theta_1$;

$$\|u\|_{L_{\alpha,V}^{\infty,\lambda}(\Omega)} \leq C \|f\|_{L_{\alpha,V}^{p,\lambda}(\Omega)}, \quad (4.3)$$

where $1 < p < n/2$, $1/p - 1/q = 2/n$, $\theta = q/p$ and $0 < \lambda < n/\theta$.

Proof. From the proof of Theorem 1.3, we have

$$\|u\|_{L_{\alpha,V}^{p,\lambda}(\Omega)} \leq C \|T_2f\|_{L_{\alpha,V}^{p,\lambda}(\Omega)} \leq C \|f\|_{L_{\alpha,V}^{p,\lambda}(\Omega)},$$

From the proof of Theorem 1.1, we obtain

$$\|D^2u\|_{L_{\alpha,V}^{p,\lambda}(\Omega)} \leq C \|D^2\mathcal{L}^{-1}f\|_{L_{\alpha,V}^{p,\lambda}(\Omega)} \leq C \|f\|_{L_{\alpha,V}^{p,\lambda}(\Omega)}.$$ 

Thus, (4.1) and (4.3) hold.

From p. 543 in [10], we know

$$|D_\alpha \Gamma'(x,y)| \leq C_k \frac{1}{(1 + |x-y|^m \mu)(x,y)} \frac{1}{|x-y|^{m-1}}, \quad (4.4)$$

where $\Gamma'(x,y)$ is the fundamental solution for $\mathcal{L} = -\Delta + V$.

Using (4.4) and adapting the argument for Theorem 1.3, we then have

$$\|Du\|_{L_{\alpha,V}^{\infty,\lambda}(\Omega)} \leq C \|D\mathcal{L}^{-1}f\|_{L_{\alpha,V}^{p,\lambda}(\Omega)} \leq C \|f\|_{L_{\alpha,V}^{p,\lambda}(\Omega)}.$$ 

Thus, (4.2) holds. Hence, Theorem 4.1 is proved. □

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References

Boundedness for some Schrödinger type operators on Morrey spaces related to certain nonnegative potentials. Article. Jul 2009. That belongs to a certain reverse Hölder class with respect to the measure $\omega(x)\,dx$. For such an operator we define the area integral $S^L_h$ associated with the heat semigroup and obtain the area integral characterization of $H^1_L$, which is the Hardy space associated with $L$. In the Euclidean setting, Tang introduced a class of Morrey spaces related to certain nonnegative potentials belonging to the reverse Hölder class and established the boundedness of some Schrödinger type operators on these new spaces in [42]. Generalized Morrey spaces and weighted Morrey spaces (including weak version) associated with Schrödinger operator were introduced and studied in [23,27,34].


7. Chen, X, Chen, J: Boundedness of sublinear operators on generalized Morrey spaces and its application. Chin. Ann. Math., Ser. A 32, 705-720 (2011). MATH Google Scholar. 13. In this paper, we first introduce some kinds of weighted Morrey spaces related to certain nonnegative potentials belonging to the reverse Hölder class $RH_q$ for $q\geq d$. Then we will establish the boundedness properties of the operators $R$ and its adjoint $R^\ast$ on these new spaces. Furthermore, weighted strong-type estimate and weighted endpoint estimate for the corresponding commutators $[b,R]$ and $[b,R^\ast]$ are also obtained. The classes of weights, the classes of symbol functions as well as weighted Morrey spaces discussed in this paper are larger than $A_p$, $\text{BMO}(\mathbb{R}^d)$ and $L^{p,\kappa}(w)$ corresponding to the classical Riesz transforms ($V\equiv0$).

Comments: 30 pages.