EXISTENCE AND UNIQUENESS RESULTS
FOR ELLIPTIC EQUATIONS WITH
COEFFICIENTS IN LOCALLY MORREY SPACES

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Abstract: In this paper we study a linear second order elliptic operator in weighted Sobolev spaces with discontinuous coefficients on unbounded domains. We prove existence and uniqueness theorems for the Dirichlet problem, when the leading coefficients $a_{ij}$ converge for $|x| \to +\infty$ and their derivatives belong to locally Morrey spaces.

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1. Introduction

In this paper we study the Dirichlet problem

\[
\begin{aligned}
Lu &= f, \\
u &\in W^2_q(\Omega) \cap W^1_q(\Omega),
\end{aligned}
\tag{1.1}
\]

for the uniformly elliptic second order linear differential operator
in an unbounded open subset $\Omega$ of $R^n$, with $f$ belonging to a weighted $L^2$-spaces denoted by $L^2_d(\Omega)$.

An existence and uniqueness theorem for problem (1.1) was proved in [5] with the following assumptions on the coefficients

\[ a \in M^t(\Omega), \]
\[ \text{essinf}_{\Omega} a > 0, \]
\[ a_{ij} \text{ convergent at infinity}, \]
\[ (a_{ij})_{x_k} \in M^0(\Omega), \quad i, j, k = 1, \ldots, n, \]
\[ a_i \in M^0(\Omega), \quad i = 1, \ldots, n, \]

where $M^t(\Omega)$ and $M^0(\Omega)$ are the so-called spaces of Morrey type defined in [9], (see Section 3 in the sequel).

In a recent paper (see [6]) we have improved the above results substituting the following hypotheses:

\[ (a_{ij})_{x_k} \in \tilde{M}^{s,n-s}_{loc}(\bar{\Omega}), \]
\[ a_i \in \tilde{M}^{s,n-s}(\bar{\Omega}), \quad i = 1, \ldots, n, \quad (1.2) \]

for some $s \in (2, n]$.

In this paper we study problem (1.1) with less restrictive assumptions, and precisely we suppose that:

\[ (a_{ij})_{x_k} \in \tilde{M}^{s,n-s}_{loc}(\bar{\Omega}), \quad \text{for some } s \in (2, n], \]
\[ \lim_{|x| \to +\infty} a_{ij} = a^0_{ij}, \quad i, j, k = 1, \ldots, n. \quad (1.3) \]

In this case we apply different technique with respect to [6]. In fact, we take an approximating sequence of operators $L_k$ for $L$ in order to establish existence and uniqueness for the perturbed problem

\[
\begin{cases}
    Lu + \lambda \beta u = f, \\
    u \in W^2_q(\Omega) \cap W^1_q(\Omega).
\end{cases}
\]

Then, we use the a priori bound of Theorem 4.3 to show the uniqueness of solution for the above problem with $\lambda = 0$ and conclude with Fredholm theory.
2. Notations and Weighted Sobolev Spaces

In this paper we use the following notations:

\[ B(x, r) = \{ y \in \mathbb{R}^n \mid |y - x| < r \}, \quad B_r = B(0, r); \]

\( \zeta_r \) is a \( C_0^\infty(\mathbb{R}^n) \) function such that:

\[ 0 \leq \zeta_r \leq 1, \quad \zeta_r|_{B_r} = 1, \quad \text{supp} \zeta_r \subset B_{2r}. \]

For a measurable subset \( E \) of \( \mathbb{R}^n \):

\( \chi_E \) is the characteristic function of \( E \)

\( \Sigma(E) \) is the Lebesgue \( \sigma \)-algebra on \( E \)

\( \mathcal{D}(E) \) is the class of restrictions to \( E \) of the functions \( \zeta \in C_0^\infty(\mathbb{R}^n) \) such that \( \text{supp} \zeta \subset E \).

Then we set

\[ |u|_{p, \Omega} = ||u||_{L^p(\Omega)}; \]

\[ u_x = \left( \sum_{i=1}^n |u_{x_i}|^2 \right)^{\frac{1}{2}}, \quad u_{xx} = \left( \sum_{i,j=1}^n |u_{x_i x_j}|^2 \right)^{\frac{1}{2}}; \]

\[ ||u||_{W^{1,p}(\Omega)} = |u|_{p, \Omega} + |u_x|_{p, \Omega}; \]

\[ ||u||_{W^{2,p}(\Omega)} = |u|_{p, \Omega} + |u_x|_{p, \Omega} + |u_{xx}|_{p, \Omega}, \]

where we omit the summability exponent if \( p = 2 \).

Now we recall the definitions of the weighted function spaces we deal with.

As in [5] we denote by \( \mathcal{A}(\Omega) \) the class of all measurable functions \( \rho : \Omega \to \mathbb{R}_+ \) such that

\[ \sup_{x \in \mathbb{B}(y, \rho(y))} \left| \log \frac{\rho(x)}{\rho(y)} \right| < +\infty, \]

i.e. for some \( \gamma \in \mathbb{R}_+ \)

\[ \gamma^{-1} \rho(y) \leq \rho(x) \leq \gamma \rho(y) \quad \forall y \in \Omega \quad \text{and} \quad \forall x \in \Omega \cap \mathbb{B}(y, \rho(y)), \quad (2.1) \]

e.g. \( \rho \) Lipschitz continuous with Lipschitz constant \(< 1 \).

We recall that a function \( \rho \in \mathcal{A}(\Omega) \) is bounded on the bounded subsets of \( \Omega \), since

\[ \rho(x) \leq a + b |x|, \quad \forall x \in \Omega, \quad (2.2) \]

with \( a \in \mathbb{R}_+ \) and \( 0 \leq b \leq 1 \).
In particular, we will employ functions \( \rho \in A(\Omega) \), which are restrictions of functions \( \rho_0 \in A(\mathbb{R}^n) \) such that

\[
\inf_{\Omega} \rho_0 > 0, \quad \lim_{|x| \to +\infty} \rho_0(x) = +\infty, \quad (2.3)
\]

e.g.

\[
\rho_0(x) = \frac{1 + |x|}{2}.
\]

For more details on weight functions we refer to Troisi [12].

Now we can define the weighted space \( L^p_q(\Omega) \), \( p \in [1, +\infty) \), \( q \in \mathbb{R} \), as the space of all measurable functions \( g : \Omega \to \mathbb{R} \), such that

\[
||g||_{L^p_q(\Omega)} = ||\rho^q g||_{p,\Omega} < +\infty.
\]

From well known results (see (26), (27) in [10]) and (2.1) we can find \( c_1, c_2 \in \mathbb{R}_+ \), such that

\[
c_1 \rho_0^{q+n}(y) \leq \int_{\mathbb{R}^n} \rho_0^q(x) \phi_a(x, y) dx \leq c_2 \rho_0^{q+n}(y), \quad (2.4)
\]

for every \( y \in \mathbb{R}^n \) and \( 0 < a \leq 1 \), where

\[
I_a(x) = \Omega \cap B(x, a \rho_0(x)),
\]

\[
\phi_a(x, y) = \begin{cases} 
1 & \text{if } y \in I_a(x), \\
0 & \text{if } y \notin I_a(x).
\end{cases}
\]

**Lemma 2.1.** Let \( p \in [1, +\infty) \), \( q \in \mathbb{R} \), \( a \in (0, 1] \). If \( g \in L^p_q(\Omega \cap B_r) \) for each \( r \in \mathbb{R}_+ \), then the following properties are equivalent:

- \( g \in L^p_q(\Omega) \);

- \( G(x) = \rho_0^{q-\frac{n}{p}}(x) |g_0|_{p,I_a(x)} \)

belongs to \( L^p(\mathbb{R}^n) \), where \( g_0 \) is the zero extension of \( g \) outside \( \Omega \). Furthermore there exist \( k_1, k_2 \in \mathbb{R}_+ \) such that

\[
k_1 ||g||_{L^p_q(\Omega)} \leq |G|_{p,\mathbb{R}^n} \leq k_2 ||g||_{L^p_q(\Omega)}.
\]
As above for $L^p_q(\Omega)$ we define the weighted Sobolev spaces $W^{k,p}_q(\Omega)$, $k = 1, 2,$ of the functions $g \in L^p_q(\Omega)$ with distributional derivatives such that
\[ ||g||_{W^{1,p}_q(\Omega)} = |\rho^q g|_{p,\Omega} + |\rho^q g_x|_{p,\Omega} < +\infty, \]
\[ ||g||_{W^{2,p}_q(\Omega)} = |\rho^q g|_{p,\Omega} + |\rho^q g_x|_{p,\Omega} + |\rho^q g_{xx}|_{p,\Omega} < +\infty, \]
where again we omit the summability exponent if $p = 2$.

$W^{k,p}_q(\Omega)$ denotes the closure of $D(\Omega)$ in $W^{k,p}_q(\Omega)$.

If $k \in \aleph_0$, we denote by $W^{k,p}_{loc}(\Omega)$ (resp. $W^{k,p}_{loc}(\Omega)$) the space of all functions $g : \Omega \to R$ such that
\[ \zeta g \in W^{k,p}_q(\Omega) \quad (\text{resp.} \quad \zeta g \in W^{k,p}_{loc}(\Omega)) \quad \forall \zeta \in D(\Omega). \]

From [5] we recall the following two results.

**Remark 2.2.** For any $\Omega$, $k \in \aleph_0$ and $q \in R$ we have
\[ W^k_q(\Omega) \subset W^k_{loc}(\Omega), \quad (2.5) \]
\[ W^k_q(\Omega) \subset W^k_{loc}(\Omega). \quad (2.6) \]

**Lemma 2.3.** If $\Omega$ has the segment property, then for any $k \in \aleph_0$ and $q \in R$ we have
\[ W^k_q(\Omega) \cap W^k_{loc}(\Omega) \subset W^k_q(\Omega). \quad (2.7) \]

### 3. The Spaces $M^{p,\lambda}$ and $M^{p,\lambda}_{loc}$

At this stage we introduce the spaces of Morrey type $M^{p,\lambda}(\Omega)$ of the functions $g \in L^p(\Omega \cap B_r)$ for each $r \in R_+$ such that
\[ ||g||_{M^{p,\lambda}(\Omega)} := \sup_{x \in \Omega, 0 < r \leq 1} |r^{-\lambda} g|_{p,\Omega \cap B(x,r)} < +\infty, \quad (3.1) \]
equipped with the norm defined in (3.1), where $0 \leq \lambda < n$ and $1 \leq p < +\infty$.

We also define the subspaces $M^{p,\lambda}_{\infty}(\Omega)$ and $M^{p,\lambda}_{0}(\Omega)$ as the closure of $L^\infty$ and $D(\Omega)$ in $M^{p,\lambda}(\Omega)$ respectively. For more details about these spaces we refer to [11].
From [6] we quote the following results, deduced from Fefferman [7] according to the version of Chiarenza et al [4].

**Theorem 3.1.** If $\Omega$ has the cone property and $g \in M^{s,n-s}(\Omega)$, $1 < p < s \leq n$, then for any $q \in \mathbb{R}$ and $u \in W^{1,p}_q(\Omega)$ we have

$$||gu||_{L^p_q(\Omega)} \leq c||g||_{M^{s,n-s}(\Omega)}||u||_{W^{1,p}_q(\Omega)}, \quad (3.2)$$

where $c \in \mathbb{R}^+$ is independent of $g$ and $u$.

**Theorem 3.2.** If $g \in M^{s,n-s}_0(\Omega)$, $1 < p < s \leq n$, in Theorem 3.1, then for any $\epsilon \in \mathbb{R}^+$ there exist $c(\epsilon) \in \mathbb{R}^+$ and an open set $\Omega_\epsilon \subset \subset \Omega$ with the cone property such that

$$||gu||_{L^p_q(\Omega)} \leq \epsilon||u||_{W^{1,p}_q(\Omega)} + c(\epsilon)||u||_{L^p_q(\Omega_\epsilon)}, \quad \forall u \in W^{1,p}_q(\Omega). \quad (3.3)$$

**Theorem 3.3.** If $\Omega$ has the cone property and $g \in M^t(\Omega)$, with $t \geq \max(2, \frac{n}{k})$, $t > 2$ if $n = 2k$, then for any $q \in \mathbb{R}$ the multiplication operator

$$u \in W^{k}_q(\Omega) \rightarrow gu \in L^2_q(\Omega)$$

is bounded, and

$$||gu||_{L^2_q(\Omega)} \leq c||g||_{M^t(\Omega)}||u||_{W^{k}_q(\Omega)}, \quad (3.4)$$

if in addition $g \in M^t_0(\Omega)$, then for any $\epsilon \in \mathbb{R}^+$ there exist $c(\epsilon) \in \mathbb{R}^+$ and an open set $\Omega_\epsilon \subset \subset \Omega$ such that

$$||gu||_{L^2_q(\Omega)} \leq \epsilon||u||_{W^{k}_q(\Omega)} + c(\epsilon)||u||_{L^2_q(\Omega_\epsilon)}, \quad (3.5)$$

and the multiplication operator is compact.

We denote by $M^{p,\lambda}_{\text{loc}}(\tilde{\Omega})$ (resp. $\tilde{M}^{p,\lambda}_{\text{loc}}(\tilde{\Omega})$) the space of the functions $g : \Omega \rightarrow \mathbb{R}$ such that

$$\zeta g \in M^{p,\lambda}(\Omega) \quad (\text{resp. } \zeta g \in \tilde{M}^{p,\lambda}(\Omega)) \quad \text{for any } \zeta \in \mathcal{D}(\tilde{\Omega}).$$

### 4. Assumptions and a Priori Bounds

Let $\Omega$ be an unbounded open subset of $\mathbb{R}^n$, $n > 2$. We assume that

- $\Omega$ has the uniform $C^2$-regularity property, according to Adams [1], i.e. there exist a locally finite open cover $\{U_j\}$ of $\partial \Omega$ and a sequence of one-to-one transformations $\phi_j \in C^2(\tilde{U}_j; \mathbb{R})$ with inverses $\psi_j \in C^2(\mathbb{R}; U_j)$ such that
− ∞
\[ \psi_j(B_{\frac{1}{2}}) \supset \Omega_{\delta} := \{ x \in \Omega, \text{dist}(x, \partial \Omega) < \delta \} \], for some \( \delta \in R_+ \).

− every collection of \( N + 1 \) of the sets \( U_j \) has empty intersection for some \( N \in \mathbb{N} \).

− \( \Phi_j(U_j \cap \Omega) = \{ (x_1, \ldots, x_n) \in B_1 : x_n > 0 \} \), for each \( j \in \mathbb{N} \).

- \( |\Phi_j(x)|, |\Phi_{j,x}(x)|, |\Phi_{j,xx}(x)| \leq M \quad \forall x \in U_j \),
  \( |\Psi_j(y)|, |\Psi_{j,y}(y)|, |\Psi_{j,yy}(y)| \leq M \quad \forall y \in B_1 \).

We consider the operator
\[
Lu = - \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} a_i u_{x_i} + au
\]

and on the coefficients of \( L \) we make the following assumption:

\[ a_{ij} = a_{ji} \in L^\infty(\Omega), \quad i, j = 1, ..., n, \quad a_i \in \tilde{M}^{s,n-s}(\Omega), \quad i = 1, ..., n, \quad \text{for some } s \in (2, n], \]

\[ (a_{ij})_{x_k} \in \tilde{M}_{loc}^{s,n-s}(\Omega), \quad i, j, k = 1, ..., n, \quad \text{for some } s \in (2, n], \]

\[ \lim_{|x| \to +\infty} a_{ij} = a_{ij}^0, \]

\( a \in \tilde{M}^t(\Omega) \), where \( t = 2 \) if \( n = 3, t > 2 \) if \( n = 4, t = \frac{n}{2} \) if \( n > 4 \). (4.5)

If \( \textbf{i}_2 \) is satisfied we denote by \( \delta : R_+ \to R_+ \) a function such that
\[
\text{esssup }_{\Omega \setminus B_r} \sum_{i,j=1}^{n} |a_{ij} - a_{ij}^0| \leq \delta(r), \quad \forall r \in R_+, \quad \lim_{r \to +\infty} \delta(r) = 0;
\]

\[ \sigma_r : \Sigma(\Omega) \to \tilde{R}_+ \text{ for any } r \in R_+ \text{ an absolutely continuous measure with respect to Lebesgue measure such that}
\]
\[ \| \chi_E \sigma_r \sum_{i,j=1}^{n} (a_{ij})_x \|_{M^{s,n-s}(\Omega)} \leq \sigma_r(E), \quad \forall E \in \Sigma(\Omega). \]
In this section we establish a priori estimates for the perturbed operator

\[ u \in W^2_q(\Omega) \rightarrow Lu + \lambda \beta u \in L^2_q(\Omega), \]

where:

i3)  \( \beta : \Omega \rightarrow R_+ \) is a function such that (see e.g. Section 3 in [6])

\[ \beta \in \tilde{M}^t(\Omega), \]  with \( t \) as in (4.5); \( \exists s \in [0, n] \) and \( \gamma \in \tilde{M}^{s, n-s}(\Omega) : \beta_x \leq \beta \gamma. \)

Lemma 4.1. If the assumptions i1), i2) and i3) are satisfied, then for any \( q, \lambda \in R_+ \) the operator

\[ u \in W^2_q(\Omega) \rightarrow Lu + \lambda \beta u \in L^2_q(\Omega) \]

is bounded.

Proof. See Lemma 3.1 of [6].

In addition we suppose that the following condition holds:

i4) there exists \( s \in [0, n] \) such that \( a_i \in M_0^{n-s}(\Omega), i = 1, \ldots, n, \) and

\[ a = a' + a'', \quad a' \in M_0^1(\Omega), \quad \operatorname{essinf}_{\Omega} a'' > 0. \]

We need the following a priori bounds from [3].

Lemma 4.2. If the assumptions i1), i2), i3), i4) are satisfied and \( \beta^{-1} \in L^\infty(\Omega \cap B_r) \) for each \( r \in R_+ \), then there exist \( \lambda_0 \in R_+ \) such that

\[ \|u\|_{W^2(\Omega)} \leq c|Lu + \lambda \beta u|_{2,\Omega}, \quad \forall u \in W^2(\Omega) \cap W^1_0(\Omega), \quad (4.6) \]

for every \( \lambda \geq \lambda_0 \) and \( c \) depending only on \( v, a^{0}_{ij}, \Omega, |a_{ij}|_{\infty, \Omega}, \delta, \sigma_r, a_i, a, n, \) and \( \beta. \)

By means of the previous result we can show the following:

Theorem 4.3. Assuming the conditions of Lemma 4.2 then, for each \( q, m \in R_+ \) and \( \lambda \geq \lambda_0, \) if we have

\[ \left\{ \begin{array}{l}
  u \in W^2_{\text{loc}}(\Omega) \cap W^1_{\text{loc}}(\Omega) \cap L^m_2(\Omega), \\
  Lu + \lambda \beta u \in L^2_q(\Omega), \end{array} \right. \quad (4.7) \]

we can deduce that \( u \in W^2_q(\Omega) \) and

\[ \|u\|_{W^2_q(\Omega)} \leq c|Lu + \lambda \beta u|_{L^2_q(\Omega)}, \quad (4.8) \]
where $c \in R_+$ is independent of $u$ and $\lambda$.

Proof. Let $r, r' \in R_+$ be such that $r < r' < 1$, and $\phi \in C_0^\infty (R^n)$ such that

$$\phi|_{B_r} = 1, \quad \text{supp} \phi \subset B_{r'},$$

$$\sup_{R^n} |\phi| \leq M, \quad \sup_{R^n} |\phi_x| \leq M(r' - r)^{-1}, \quad \sup_{R^n} |\phi_{xx}| \leq M(r' - r)^{-2}.$$

For each $y \in R^n$ we define

$$\psi(x) := \phi\left(\frac{x - y}{\rho_0(y)}\right)$$

and so we can apply (4.6) to get the local estimate

$${||\psi u||}_{W_2^2(\Omega)} \leq c_0|L(\psi u) + \lambda \beta \psi u|_{2, \Omega}.$$  \hfill (4.9)

As in [6], using (2.3), (3.2), standard interpolation inequalities and a well known monotonicity result of Miranda [8, Lemma 3.1 ] we obtain

$${||u||}_{W_2^2(I_{1/2}(y))} \leq c[|L u + \lambda \beta u|_{2, I_{1/2}(y)} + \rho_0^{-1}(y)|u|_{2, I_{1/2}(y)}],$$  \hfill (4.10)

whence, integrating on $R^n$, from Lemma 2.1 we deduce

$${||u||}_{W_2^2(\Omega)} \leq c(|L u + \lambda \beta u|_{L_2^2(\Omega)} + |u|_{L_2^{-2}(\Omega)}).$$  \hfill (4.11)

So arguing as in the proof of Theorem 4.1 of [5], then from (4.7) we deduce that $u \in W_2^2(\Omega)$ and from (3.3) or (3.5), since $\rho \in L_\infty^\infty(\Omega)$ by (2.2), we have

$${||u||}_{W_2^2(\Omega)} \leq c(|L u + \lambda \beta u|_{L_2^2(\Omega)} + |u|_{2, \Omega_0}),$$  \hfill (4.12)

where $\Omega_0$ is an open set $\subset \subset \Omega$.

Finally, proceeding along the same lines of Viola [13] (see also [5], Corollary 4.1), by means of the boundedness of $L$ and $\beta^{-1} \in L_\infty^\infty(\Omega \cap B_r)$ for each $r \in R_+$, we get inequality (4.8). \hfill \Box

5. Existence and Uniqueness Results

Now we are in position to state the following existence result for the perturbed problem with large values of $\lambda$. 

\hfill \Box
Theorem 5.1. Assuming the conditions of Lemma 4.2, for any \( q \in \mathbb{R} \) there exists \( \lambda_1 \in \mathbb{R}_+ \) such that the Dirichlet problem

\[
\begin{cases}
Lu + \lambda \beta u = f, \\
u \in W^2_q(\Omega) \cap W^1_q(\Omega),
\end{cases}
\]  

(5.1)

is uniquely solvable for every \( f \in L^2_q(\Omega) \) and \( \lambda \geq \max(\lambda_0, \lambda_1) \).

Proof. We consider the operator

\[ L_k u = - \sum_{i,j=1}^{n} a_{ij}^k u_{x_i x_j} + \sum_{i=1}^{n} a_i u_{x_i} + au, \]

where the \( a_{ij}^k \) are defined as following. We set, for any \( i, j = 1, \ldots, n, \)

\[ \tilde{a}_{ij} = \begin{cases} a_{ij} & \text{in } \Omega \\ a_{ij}^0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \]

Let \((J_k)_{k \in \mathbb{N}}\) be a sequence of mollifiers and, for any \( k \in \mathbb{N} \), we define

\[ a_{ij}^k = \zeta_k \tilde{a}_{ij} + (1 - \zeta_k) J_k * \tilde{a}_{ij}. \]

Obviously, \( a_{ij}^k \in L^\infty(\Omega) \). Furthermore, since \( \lim_{|x| \to +\infty} a_{ij}(x) = a_{ij}^0 \), then the quadratic form \( \sum_{i,j=1}^{n} a_{ij}^k(x) \xi_i \xi_j \) is positive definite, \( a_{ij}^k \) converge to \( a_{ij}^0 \), when \( |x| \to +\infty \) and

\[
\lim_{|x| \to +\infty} (a_{ij}^k)_{x_h}(x) = \lim_{|x| \to +\infty} \int_{\Omega \cap B(x, \frac{1}{k})} (J_k)_{x_h}(x - y)(a_{ij}(y) - a_{ij}^0)dy = 0.
\]

From the last equality and Section 2 of [11] we deduce that \((a_{ij}^k)_{x_h} \in M_0^{s,n-s}(\Omega).\) Therefore choosing \( \lambda \geq \lambda_1 \) as in Theorem 4.1 of [6], the problem

\[ u \in W^2_q(\Omega) \cap W^1_q(\Omega), \quad L_k u + \lambda \beta u = f, \quad f \in L^2_q(\Omega) \]

(5.2)

admits an unique solution \( u_k \).

Moreover, an inequality of type (4.7) holds for any \( u_k \), i.e.

\[
\|u_k\|_{W^2_q(\Omega)} \leq c\|f\|_{L^2_q(\Omega)},
\]

(5.3)
with a constant $c$ independent of $k$.

In fact we have for any $k \in \mathbb{N}$

a) \[
\sum_{i,j=1}^{n} a_{ij}^k \xi_i \xi_j \geq \frac{\nu}{2} |\xi|^2, \quad \text{a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}^n;
\]

b) \[
\lim_{|x| \to +\infty} a_{ij}^k(x) = a_{ij}^0;
\]

c) \[
\text{esssup}_{\Omega} |a_{ij}^k| \leq ||a_{ij}||_{L^\infty(\Omega)};
\]

d) there exists $c \in \mathbb{R}_+$ such that \[
\text{esssup}_{\Omega \setminus B_{r+1}} \sum_{i,j=1}^{n} |a_{ij}^k - a_{ij}^0| \leq c\delta(r), \quad \forall r \in \mathbb{R}_+;
\]

e) we have \[
||\chi \varepsilon \zeta r \sum_{i,j=1}^{n} (a_{ij}^k)_x||_{M^{s,n-s}(\Omega)} = ||\chi \varepsilon \zeta r \sum_{i,j=1}^{n} (a_{ij})_x||_{M^{s,n-s}(\Omega)} \leq \sigma_r(E),
\]

for any $E \in \Sigma(\Omega)$ provided that $k \geq 2r$ and $r \in \mathbb{R}_+$.

Since $W^2_q(\Omega)$ is a reflexive Banach space (see [2]), from (5.3) it follows that there exists a subsequence $(u_{k_n})_{k \in \mathbb{N}}$ weakly convergent in $W^2_q(\Omega)$ to a function $u \in W^2_q(\Omega) \cap W^1_q(\Omega)$. Therefore, we have \[
L_k u_{k_n} + \lambda \beta u_{k_n} \rightharpoonup L u + \lambda \beta u,
\]
in the sense of distributions, whence the existence of solution is proved. The uniqueness result follows from Theorem 4.3 when $\lambda \geq \lambda_0$.

**Theorem 5.2.** Suppose that conditions of Lemma 4.2 hold, the operator

\[
u \in \mathbb{R} \quad \text{and} \quad \lambda \in \mathbb{R}
\]

is a Fredholm operator with index equal to zero.
Proof. Choosing \( \beta(x) = \frac{1}{1+|x|^2} \), we can satisfy the assumption of Lemma 4.2 and so apply Theorem 5.1 to deduce that the problem

\[
\begin{cases}
Lu + \lambda_2 \beta u = f, \\
u \in W^2_q(\Omega) \cap W^1_1(\Omega),
\end{cases}
\]

is uniquely solvable for every \( f \in L^2_q(\Omega) \), with \( \lambda_2 \geq \lambda_1 \).

But \( \beta \in M^t_0(\Omega) \), with \( t \) as required by Theorem 3.3, and so the operator

\[
u \in W^2_q(\Omega) \to \lambda_2 \beta u \in L^2_q(\Omega)
\]

is compact, whence a classical argument of Fredholm theory concludes the proof.

\[\blacksquare\]

**Theorem 5.3.** Suppose that the conditions (i1), (i2) and (i4) hold, together with the assumption \( \text{essinf}_{\Omega} a > 0 \).

Then for each \( q \in \mathbb{R} \) the problem

\[
\begin{cases}
Lu = f, \\
u \in W^2_q(\Omega) \cap W^1_1(\Omega),
\end{cases}
\]

is uniquely solvable for every \( f \in L^2_q(\Omega) \).

Proof. By Theorem 5.2, and by virtue of Fredholm theory it is sufficient to show the uniqueness.

For this purpose, let \( u \in W^2_q(\Omega) \cap W^1_1(\Omega) \) be such that \( Lu = 0 \). By using Theorem 4.3 and arguing as in Theorem 4.3 in [6], we obtain that \( u \in W^2_q(\Omega) \cap W^1_1(\Omega) \). From Theorem 5.2 of [3] we deduce that \( u = 0 \) since it is unique solution of the homogeneous Dirichlet problem

\[
\begin{cases}
Lu = 0, \\
u \in W^2(\Omega) \cap W^1(\Omega).
\end{cases}
\]

The proof is completed. \[\blacksquare\]

**References**

EXISTENCE AND UNIQUENESS RESULTS...


existence and uniqueness of weak energy solutions of the Dirichlet problem in anisotropic Sobolev-Orlicz spaces prompted by the equation. \( \alpha \) and \( \beta \) boundedness of energy solutions, \( \alpha \) and \( \beta \) localization (vanishing) properties of energy solutions. We study the Dirichlet problem with zero boundary conditions for the elliptic equations with variable anisotropic nonlinearities. Xn \( \alpha \) and \( \beta \) \( \sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{m} a_i(x,u)\left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} = f(x,u) \) in a bounded domain \( \alpha \) and \( \beta \) with Lipschitz-continuous boundary \( \Gamma = \partial \Omega \). It is assumed that \( p_i(x) = a_i(x) \), \( \beta \) and \( \gamma \) \( a_i(x) \) \( C^0(\Omega) \) with logarithmic module of continuity. Uniqueness Results for Higher Order Elliptic Equations in Weighted Sobolev Spaces. Int. J. Differ. P. Di Gironimo and A. Vitolo. \( \alpha \) and \( \beta \) Existence and uniqueness results for elliptic equations with coefficients in locally Morrey spaces, \( \alpha \) and \( \beta \) International Journal of Pure and Applied Mathematics, vol. 4, pp. 181–194, 2003. MR1956879. Mathematical Reviews (MathSciNet): MR1956879. S. Monsurrö and M. Transirico, \( \alpha \) and \( \beta \) Dirichlet problem for divergence form elliptic equations with discontinuous coefficients, \( \alpha \) and \( \beta \) Boundary Value Problems, vol. 2012, article 67, 2012. Zentralblatt MATH: 1278.35065 Digital Object Identifier: doi:10.1186/1687-2770-2012-67. This paper deals with an existence and uniqueness result of a weak solution for a quasilinear elliptic boundary value problem in a domain whose boundary is the union of two disjoint closed surfaces. On the interior boundary we prescribe a nonlinear Robin condition with suitable growth assumptions, and on the exterior boundary, a Dirichlet condition. The main difficulty when proving the existence of a solution is due to the nonlinear boundary condition, since, in order to apply a fixed point theorem, we need to prove the weak continuity of the associated boundary operator. To this aim, we first study several properties of this operator. Issue no: Vol 27/2011 no. 2 Tags: \( \alpha \) and \( \beta \) Quasilinear elliptic equations, nonlinear boundary conditions. In physics by the Navier-Stokes equations is meant the impulse equation for the flow. In the computational fluid dynamics the impulse equation is enlarged by the continuity and energy equations. \( \alpha \) and \( \beta \) However, in this full generality no uniqueness theorem for a weak solution has been known. On the other hand, under stronger conditions on the solution, it is unique, cf. [17,18]. Yet another motivation consists in the study of anisotropic elliptic problems, for example, parabolic problems which include the first mixed problem for the heat equation in a cylinder CT. For a nonnegative integer \( s \), the norm of \( H^{2s}(CT, F1) \) controls the derivatives \( d^{a}d^{j}u \) with \( \alpha + 2j \) in the \( L_2 \)-norm on CT.